# AN INTRODUCTION TO NUMERICAL METHODS USING MATHCAD

## Mathcad Release 14

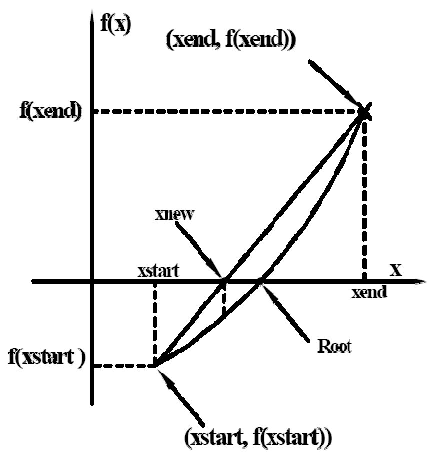
$h := 1..5$

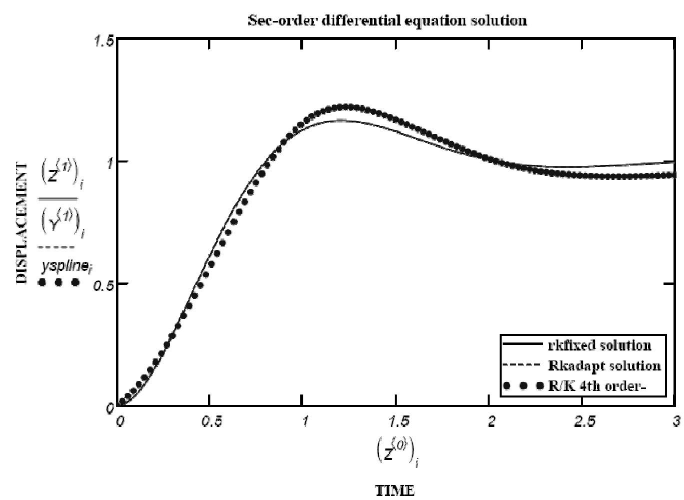
$x := 0, 1..5$

Various orders of approximation generated by Taylor series approach versus true values of given function

| h = |
|---|
| 1 |
| 2 |
| 3 |
| 4 |
| 5 |

| | zeroth order | first order | second order | third order | fourth order | fifth order | True Value |
|---|---|---|---|---|---|---|---|
| $x =$ | $ftaylor0(x)$ | $ftaylor1(x) =$ | $ftaylor2(x)$ | $ftaylor3(x) =$ | $ftaylor4(x) =$ | $ftaylor5(x) =$ | $f(x) =$ |
| 0 | 1.25 | 1.25 | 1.25 | 1.25 | 1.25 | 1.25 | 1.25 |
| 1 | 1.25 | 1 | 0.75 | 0.58 | 0.43 | 0.43 | 0.43 |
| 2 | 1.25 | 0.75 | -0.25 | -1.61 | -4.01 | -4.01 | -4.01 |
| 3 | 1.25 | 0.5 | -1.75 | -6.34 | -18.49 | -18.49 | -18.49 |
| 4 | 1.25 | 0.25 | -3.75 | -14.63 | -53.03 | -53.03 | -53.03 |
| 5 | 1.25 | 0 | -6.25 | -27.5 | -121.25 | -121.25 | -121.25 |





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# C H A P T E R 2

# INTRODUCTION TO NUMERICAL METHODS

## 2.1 THE USE OF NUMERICAL METHODS IN SCIENCE AND ENGINEERING

Analysis of problems in engineering and the physical sciences typically involves four steps as follows.

(1) Development of a suitable mathematical model that realistically represents a given physical system.

(2) Derivation of the system governing equations using physical laws such as Newton's laws of motion, conservation of energy, the laws governing electrical circuits etc.

(3) Solution of the governing equations, and

(4) Interpretation of the results.

Because real world problems are generally quite complex with the generation of closed-form analytical solutions becoming impossible in many situations, there exists, most definitely, a need for the proper utilization of computer-based techniques in the solution of practical problems. The advancement of computer technology has made the effective use of numerical methods and computer-based techniques very feasible, and thus, solutions can now be obtained much faster than ever before and with much better than acceptable accuracy. However, there are advantages as well as disadvantages associated with any numerical procedure that is resorted to , and these must be kept in mind when using it.

## 2.2 COMPARISON OF NUMERICAL METHODS WITH ANALYTICAL METHODS

While an analytical solution will be exact if it exists, a numerical method , on the other hand, will generally require iterations to generate a solution, which is only an *approximation* and which certainly cannot be considered *exact* by any means.

A disadvantage associated with analytical solution techniques is that they are generally applicable only to very special cases of problems. Numerical solutions, on the contrary, will solve complex situations as well.

While numerical techniques have several advantages including easy programming on a computer and the convenience with which they handle complex problems, the initial estimate of the solution along with the many number of iterations that are sometimes required to generate a solution can be looked upon as disadvantages.

## 2.3 SOURCES OF NUMERICAL ERRORS AND THEIR COMPUTATION

It is indeed possible for miscalculations to creep into a numerical solution because of various sources of error. These include inaccurate mathematical modeling, wrong programming, wrong input, rounding off of numbers and truncation of an infinite series. Round-off error is the general name given to inaccuracies that affect the calculation scene when a finite number of digits are assigned to represent an actual number. In a long sequence of calculations, this round-off error can accumulate, then propagate through the process of calculation and finally grow very rapidly to a significant number. A truncation error results when an infinite series is approximated by a finite number of terms, and, typically, upper bounds are placed on the size of this error.

The true error is defined as the difference between the computed value and the true value of a number.

$$E_{True} = X_{Comp} - X_{True} \qquad (2.1)$$

while the relative true error is the error relative to the true value

$$e_r = \frac{X_{Comp} - X_{True}}{X_{True}} \qquad (2.2)$$

Expressed as a percentage, the relative true error is written as

$$e_r = \frac{X_{Comp} - X_{True}}{X_{True}} \cdot 100 \qquad (\%) \qquad (2.3)$$

## 2.4 TAYLOR SERIES EXPANSION

The Taylor series is considered as a basis of approximation in numerical analysis. If the value of a function of $x$ is provided at "$x_0$", then the Taylor series provides a means of evaluating the function at "$x_0 + h$", where "$x_0$" is the starting value of the independent variable and "$h$" is the difference between the starting value and the new value at which the function is to be approximated

$$f(x_0 + h) = f(x_0) + h \cdot \left[\frac{d}{dx}\left[f(x_0)\right]\right] + \frac{h^2}{2!} \cdot \frac{d^2}{dx^2} f(x_0) + \frac{h^3}{3!} \cdot \frac{d^3}{dx^3} f(x_0) + \ldots\ldots\ldots\ldots \qquad (2.4)$$

This equation can be used for generating various orders of approximations as shown below. The order of approximation is defined by the highest derivative included in the series. For example, If only terms up to the second derivative are retained in the series, the result is a second order approximation.

<u>First order approximation:</u>

$$f(x_0 + h) = f(x_0) + h\cdot\frac{d}{dx}f(x_0) \qquad (2.5)$$

<u>Second order approximation:</u>

$$f(x_0 + h) = f(x_0) + h\cdot\frac{d}{dx}f(x_0)\cdot\frac{d^2}{dx^2}f(x_0) \qquad (2.6)$$

<u>Third order approximation:</u>

$$f(x_0 + h) = f(x_0) + h\cdot\frac{d}{dx}f(x_0) + \frac{h^2}{2!}\cdot\frac{d^2}{dx^2}f(x_0) + \frac{h^3}{3!}\cdot\frac{d^3}{dx^3}f(x_0) + \ldots\ldots\ldots\ldots \qquad (2.7)$$

It is to noted that the significance of the higher order terms in the Taylor series increases with the nonlinearity of the function involved as well as the difference between the *" starting x"* value and the *"x"* value at which the function is to be approximated. Thus , the fewer the terms that are included in the series, the larger will be the error associated with the computation of the function value. If the function is linear , however, only terms up to the first derivative term need to be included.

**Example 2.1**

Using the Taylor series expansion for

$$f(x) = -0.15\,x^4 - 0.17\,x^3 - 0.25\,x^2 - 0.25\,x + 1.25$$

determine the zeroth, first, second, third, fourth and fifth order approximations of $f(x0 + h)$ where $x0 = 0$ and $h = 1,2, 3, 4,5$ and compare these with the exact solutions.

*<u>h=1.0:</u>* Put in the function and generate its derivatives:

$$f(x) := -0.15\cdot x^4 - 0.17\cdot x^3 - 0.25\cdot x^2 - 0.25x + 1x0 := 0 \quad h := 1.$$

$$fprime(x) := -0.60\cdot x^3 - 0.51\cdot x^2 - 0.50\cdot x - 0.25$$

<--Generate derivatives

$$f2prime(x) := -1.8\cdot x^2 - 1.02\cdot x - 0.50$$

$$f3prime(x) := -3.6\cdot x - 1.02 \quad f4prime(x) := -3.6 \quad f5prime(x) := 0.$$

$term1 := f(x0) \quad term2 := h \cdot fprime(x0) \quad term3 := \frac{h^2}{2} \cdot f2prime(x0)$

$term4 := \frac{h^3}{6} \cdot f3prime(x0) \quad term5 := \frac{h^4}{24} \cdot f4prime(x0) \quad term6 := \frac{h^5}{120} \cdot f5prime(x0)$

$ftaylor0 := term1$ <---- one-term or zero-order approximation

$ftaylor1 := term1 + term2$ <---- first order approximation with two terms

$ftaylor2 := term1 + term2 + term3$ <---second order approximation with 3 terms

$ftaylor3 := term1 + term2 + term3 + term4$ <---third order approximation with 4 terms

$ftaylor4 := term1 + term2 + term3 + term4 + term5$ <---fourth order approximation with 5 terms

$ftaylor5 := term1 + term2 + term3 + term4 + term5 + term6$ <----- fifth order approximation with 6 terms

$x := x0 + h \quad x = 1$

$ftaylor0 = 1.25 \quad ftaylor1 = 1 \quad ftaylor2 = 0.75$
$ftaylor3 = 0.58 \quad ftaylor4 = 0.43 \quad ftaylor5 = 0.43$ <-- These are the zero- fifth order approximations of the given function f(x) using the Taylor series.

$f(1) = 0.43$ <---EXACT ANSWER USING FUNCTION GIVEN.

$err0 := f(x) - ftaylor0 \quad err0 = -0.82$

$err1 := f(x) - ftaylor1 \quad err1 = -0.57$

$err2 := f(x) - ftaylor2 \quad err2 = -0.32$

$err3 := f(x) - ftaylor3 \quad err3 = -0.15$

**<--** These are errors ( differences between exact values and approximations ) for the above zero - fifth order approximations.

$err4 := f(x) - ftaylor4 \quad err4 = 0$

$err5 := f(x) - ftaylor5 \quad err5 = 0$

Similarly, by using *h= 2, 3, 4, 5* , the zeroth- fifth order approximations for *f(2), f(3), f(4), f(5)* and the associated errors can be determined. These are given in Tables 2.1 and 2.2

Plots of the various Taylor series approximations of the given function and associated errors are generated below and are presented in Figs. 2.1 and 2.2

$x0 := 0 \qquad x := 0, 0.01 .. 5$

$ftaylor0(x) := f(x0)$ <-- zeroth-order approximation

$ftaylor1(x) := ftaylor0(x) + (x - x0) \cdot fprime(x0)$ <---first-order approximation

$ftaylor2(x) := ftaylor1(x) + \frac{(x - x0)^2}{2} \cdot f2prime(x0)$ <--second-order approximation

$ftaylor3(x) := ftaylor2(x) + \frac{(x - x0)^3}{6} \cdot f3prime(x0)$ <---third-order approximation

$ftaylor4(x) := ftaylor3(x) + \frac{(x - x0)^4}{24} \cdot f4prime(x0)$ <---fourth-order approximation

$ftaylor5(x) := ftaylor4(x) + \frac{(x - x0)^5}{120} \cdot f5prime(x0)$ <---fifth- order approximatio**n**

Errors generated with the various approximations are as follows

Zero order approximation: $err0(x) := f(x) - ftaylor0(x)$

First order approximation: $err1(x) := f(x) - ftaylor1(x)$

Second order approximation: $err2(x) := f(x) - ftaylor2(x)$

Third order approximation: $err3(x) := f(x) - ftaylor3(x)$

Fourth order approximation: $err4(x) := f(x) - ftaylor4(x)$

Fifth order approximation: $err5(x) := f(x) - ftaylor5(x)$

The various approximations generated by the above calculations and the associated errors are compared in the Table 2.1.

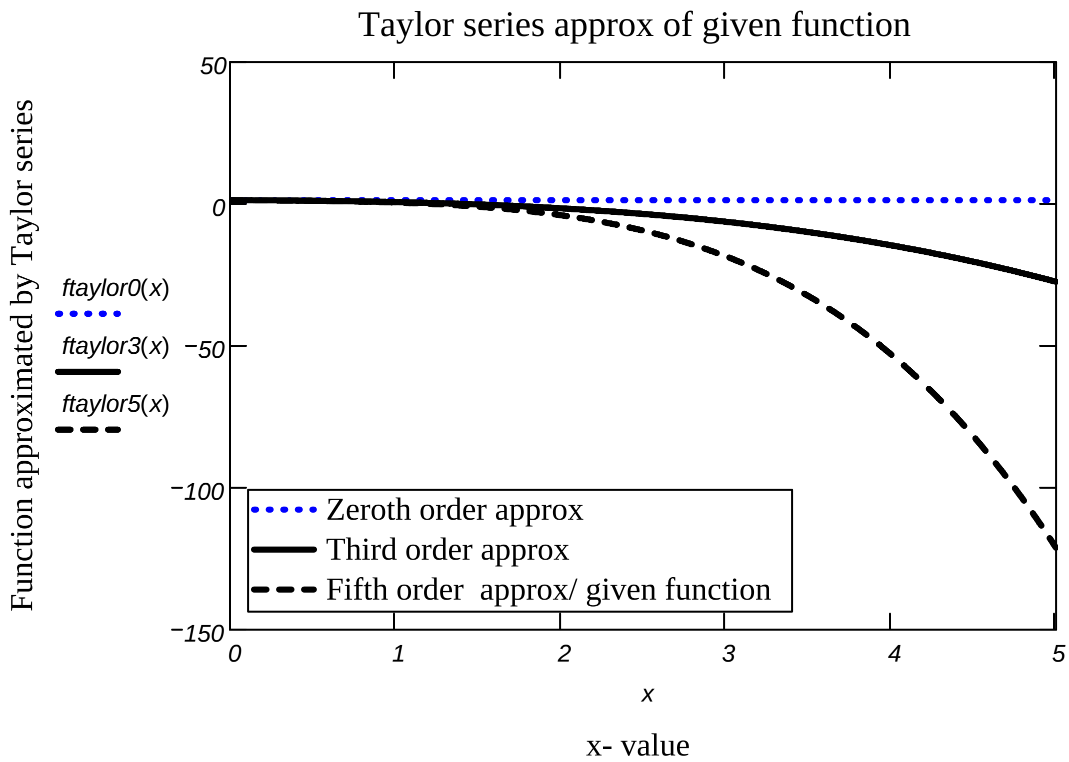


**Figure 2.1.** Taylor series approximation of given function

$x := 0, 0.01 .. 5$

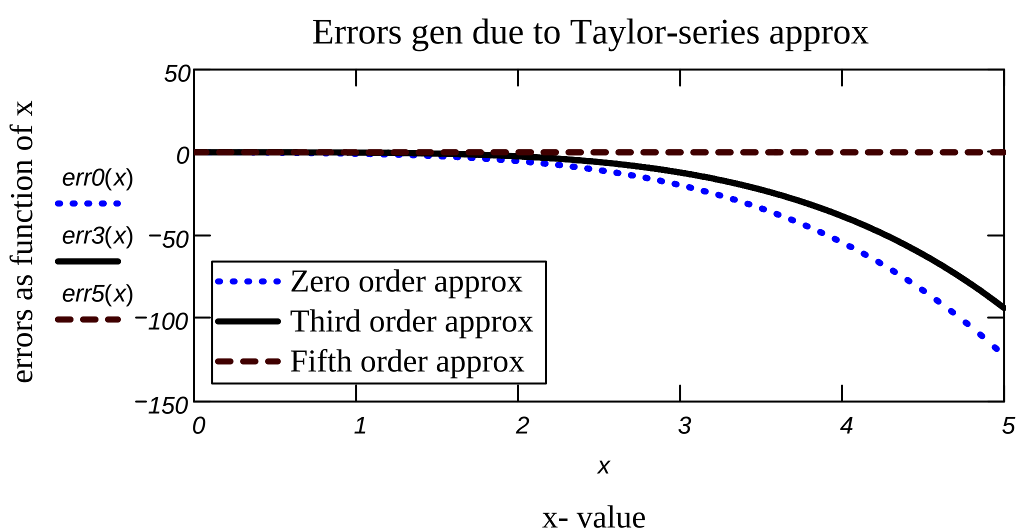


**Figure 2.2.** Errors associated with the various Taylor series approximations

The various approximations generated by the above calculations and the associated errors are compared in the following tables.

**Table 2.1**

$h := 1 .. 5$

$x := 0, 1 .. 5$

Various orders of approximation generated by Taylor series approach versus true values of given function

| $h =$ | $x =$ | zeroth order $ftaylor0(x)$ | first order $ftaylor1(x)$ | second order $ftaylor2(x)$ | third order $ftaylor3(x)$ | fourth order $ftaylor4(x)$ | fifth order $ftaylor5(x)$ | True Value $f(x) =$ |
|---|---|---|---|---|---|---|---|---|
| | 0 | 1.25 | 1.25 | 1.25 | 1.25 | 1.25 | 1.25 | 1.25 |
| 1 | 1 | 1.25 | 1 | 0.75 | 0.58 | 0.43 | 0.43 | 0.43 |
| 2 | 2 | 1.25 | 0.75 | -0.25 | -1.61 | -4.01 | -4.01 | -4.01 |
| 3 | 3 | 1.25 | 0.5 | -1.75 | -6.34 | -18.49 | -18.49 | -18.49 |
| 4 | 4 | 1.25 | 0.25 | -3.75 | -14.63 | -53.03 | -53.03 | -53.03 |
| 5 | 5 | 1.25 | 0 | -6.25 | -27.5 | -121.25 | -121.25 | -121.25 |

**Table 2.2**

Errors associated with the different orders of approximation

| $h =$ | $x =$ | zeroth order $err0(x) =$ | first order $err1(x) =$ | second order $err2(x) =$ | third order $err3(x) =$ | fourth order $err4(x) =$ | fifth order $err5(x) =$ |
|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | -0.82 | -0.57 | -0.32 | -0.15 | 0 | 0 |
| 2 | 2 | -5.26 | -4.76 | -3.76 | -2.4 | 0 | 0 |
| 3 | 3 | -19.74 | -18.99 | -16.74 | -12.15 | 0 | 0 |
| 4 | 4 | -54.28 | -53.28 | -49.28 | -38.4 | 0 | 0 |
| 5 | 5 | -122.5 | -121.25 | -115 | -93.75 | 0 | 0 |

## PROBLEMS

2.1. Using the Taylor series expansion for $\cos x$, which is given as

$$f(x) = \cos x = 1 - x^2/2 + x^4/24 ,$$

determine the one-term, two-term and three-term approximations of $f(x_0 + h)$, where $x_0 = 0$

*rad* and $h = 0.1, 0.2 ....1.0$ *rad*, and compare these with the exact solution. Using Mathcad, generate plots of the various Taylor series approximations and associated errors as functions of the independent variable $x$.

2.2 Develop a Taylor series expansion of the following function:

$$f(x) = x^5 - 6x^4 + 3x^2 + 9 .$$

Use $x = 3$ as the base and h as the increment. Using Mathcad, evaluate the series for $h = 0.1, 0.2....1.0,$ adding terms incrementally as in Problem 2.1. Compare the various Taylor series approximations obtained with true values in a table. Generate plots of the approximations and associated errors as functions of $x$ .

2.3 Given the following function:

$$f(x) = x^3 - 3x^2 + 5x + 10 ,$$

determine $f(x_0 + h)$ with the help of a Taylor series expansion, where $x_0 = 2$ and $h = 0.4.$ Compare the true value of $f(2.4)$ with estimates obtained by resorting to (a) one term only (b) two terms (c) three terms and (d) four terms of the series.

2.4 Given the following function

$$f(x) = 3 \cdot x^3 - 6 \cdot x^2 + 15 \cdot x + 25$$

use a Taylor series expansion to determine the zeroth, first, second and third order approximations of $f(x_0+h)$ where $x_0 = 2$ and $h = 0.5$. Compare these with the exact solution.

2.5 By developing a Taylor series expansion for

$$f(x) = e^x$$

about $x = 0$ , determine the fourth-order approximation of $e^{2.5}$ and compare it with the exact solution.

2.6. By developing a Taylor series expansion for

$$f(x) = \ln(2-x)$$

about $x = 0$, determine the fourth-order approximation of $\ln(0.5)$ and compare it with the exact solution

2.7. By developing a Taylor series expansion for

$$f(x) = x^3 e^{-5x}$$

about $x = 1$, determine the third-order approximation of $f(1.2)$ and compare it with the exact solution.

2.8. By developing a Taylor series expansion for

$$f(x) = e^{\cos x}$$

about $x = 0$ , determine the fourth- order approximation of $f(2\pi)$ and compare it with the exact solution. .

2.9. By developing a Taylor series expansion for

$$f(x) = (x - 2)^{1/2}$$

about $x = 2$, determine the third-order approximation of $f(2.2)$, that is, $(0.2)^{1/2}$, and compare it with the exact solution.

2.10. Given the function

$$f(x) = x^2 - 5x^{0.5} + 6 ,$$

use a Taylor series expansion to determine the first, second , third and fourth order approximations of $f(2.5)$ by resorting to $x_0 = 2$ and $h = 0.5$ . Compare these with the exact solution.

2.11. Given the function

$$f(x) = 6x^3 - 9x^2 + 25x + 40 ,$$

use a Taylor series expansion to determine the zeroth, first, second and third order approximations of $f(x_0 + h)$ where $x_0 = 3$ and $h = 1$ . Compare these with the exact solution.

2.12. Given the function

$$f(x) = 4x^4 - 7x^3 + 5x^2 - 6x + 90$$

use a Taylor series expansion to determine the zeroth, first, second , third and fourth order *approximations of* $f(x_0 + h))$ where $x_0 = 3$ and $h = 0.5$ . Compare these with the exact solution. Calculate errors abnd generate calculations to three decimal places.

2.13. Given the function

$$f(x) = 8x^3 - 10x^2 + 25x + 45 \quad ,$$

use a Taylor series expansion to determine the zeroth, first, second and third order approximations of $f(x_0 + h)$ where $x_0 = 2$ and $h = 1$ . Compare these with the exact solution.

2.14. Given the function

$$f(x) = 1 + x + x^2 / 2! + x^3 / 3! + x^4 / 4!$$

use a Taylor series expansion to determine the zeroth, first, second , third and fourth order approximations of $f(x_0 + h)$ where $x_0 = 0$ and $h = 0.5$ . Compare these with the exact solution. Generate answers correct to four decimal places.

2.15. Given the function

$$f(x) = x + x^3 / 3 + 2x^5 / 15$$

use a Taylor series expansion to determine the zeroth, first, second , third and fourth order approximations of $f(x_0 + h)$ where $x_0 = 0$ and $h = 0.8$ . Compare these with the exact solution by computing percentage errors. Generate answers correct to four decimal places.

2.16. Given the function

$$f(x) = \sin(x)$$

use a Taylor series expansion to determine the fifth order approximation of $f(x_0 + h)$ where $x_0 = 0$ and $h = 0.2$ *radians* . Compare your answer with the true value. Generate answers correct to four decimal places.

2.17. Given the function

$$f(x) = 3x^2 - 6x^{0.5} + 9 \, ,$$

use a Taylor series expansion to determine the zeroth, first, second , third and fourth order approximations of $f(x_0 + h)$ whereo $x_0 = 3$ and $h = 1$ . Compare these with the exact solution by computing percentage errors. Generate answers correct to four decimal places.